Fill Ups of Mathematical Induction and Binomial Theorem

Q.1. The larger of $99^{50} + 100^{50}$ and 101^{50} is .... (1982 - 2 Marks)

Ans. $(101)^{50}$

Sol. Consider $(101)^{50} - \{(99)^{50} + (100)^{50}\} = (100 + 1)^{50} - (100 - 1)^{50} - (100)^{50}$

$= (100)^{50} [(1 + 0.0\ 1)^{50} - (1 - 0.01)^{50} - 1]$

$= (100)^{50} [2\ (^{50}C_1(0.01) + {}^{50}C_3(0.01)^3 + ...) - 1]$

$= (100)^{50} [2\ (^{50}C_3(0.01)^3 + ...)] > 0$

$\therefore (101)^{50} > (99)^{50} + (100)^{50}$

$\therefore (101)^{50}$ is greater.

Q.2. The sum of the coefficients of the plynomial $(1 + x - 3x^2)^{2163}$ is .... (1982 - 2 Marks)

Ans. -1

Sol. If we put $x = 1$ in the expansion of $(1 + x - 3x2)^{2163} = A_0 + A_1x + A_2x^2 + ...$
we will get the sum of coefficients of given polynomial, which clearly comes to be $- 1$.

Q.3. If $(1 + ax)^n = 1 + 8x + 24x^2 + ...$ then a = ... and n = ... (1983 - 2 Marks)

Ans. a = 2, n = 4

Sol. $(1 + ax)^n = 1 + 8x + 24x^2 + ...$

$\Rightarrow (1 + ax)^n = 1 + nxa + \dfrac{n(n-1)}{2!}.a^2x^2 + ...$

$= 1 + 8\ x + 24x\ 2 + ...$

Comparing like powers of x we get

$nax = 8x \Rightarrow na = 8$ ....(1)

$\dfrac{n(n-1)a^2}{2} = 24 \quad \Rightarrow \quad n(n-1)a^2 = 48$ ....(2)

Solving (1) and (2), $n = 4$, $a = 2$

Q.4. Let n be positive integer. If the coefficients of 2nd, 3rd, and 4th terms in the expansion of $(1 + x)^n$ are in A.P., then the value of n is .... (1994 - 2 Marks)

Ans. 7

Sol. We know that for a +ve integer n $(1 + x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + ...+ {}^nC_n x^n$
ATQ coefficients of 2nd, 3nd, and 4th terms are in A.P. i.e.${}^nC_1$, ${}^nC_2$, ${}^nC_3$ are in A.P.
$\Rightarrow 2.{}^nC_2 = {}^nC_1 + {}^nC_3$

$\Rightarrow \quad 2 \times \dfrac{n(n-1)}{2} = n + \dfrac{n(n-1)(n-2)}{3!}$

$\Rightarrow \quad n - 1 = 1 + \dfrac{n^2 - 3n + 2}{6} \Rightarrow n^2 - 9n + 14 = 0$

$\Rightarrow (n - 7)(n - 2) = 0 \Rightarrow n = 7$ or 2

But for the existance of 4th term, n = 7.

**Q.5. The sum of the rational terms in the expansion of $(\sqrt{2} + 3^{1/5})^{10}$ is ....
(1997 - 2 Marks)**

Ans. 41

Sol. Let T_{r+1} be the general term in the expansion of

$(\sqrt{2} + 3^{1/5})^{10}$

$\therefore \quad T_{r+1} = {}^{10}C_r (\sqrt{2})^{10-r} . (3^{1/5})^r . (0 \le r \le 10)$
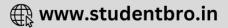
$= \dfrac{10!}{r!(10-r)!} . 2^{5-r/2} 3^{r/5}$

Let T_{r+1} will be rational if $2^{5-r/2}$ and $3^{r/5}$ are rational numbers.

$\Rightarrow 5-\dfrac{r}{2}$ and $\dfrac{r}{5}$ are integers.

$\Rightarrow r = 0$ and $r = 10$ $\Rightarrow T_1$ and T_{11} are rational terms.

$\Rightarrow$ Sum of T_1 and $T_{11} = {}^{10}C_0 2^5 - 0.3^0 + {}^{10}C_{10} 2^{5-5}.3^2$

$= 1.32.1 + 1.1.9 = 32 + 9 = 41$

Subjective questions of Mathematical Induction and Binomial Theorem

Q. 1. Given that (1979) $C_1 + {}^2C_2x + {}^3C_3x^2 + \ldots\ldots\ldots + 2n\,C_{2n}x^{2n-1} = 2n\,(1+x)^{2n-1}$

where Cr = $\dfrac{(2n)!}{r!(2n-r)!}$ $\qquad$ **r = 0, 1, 2, ..., 2n**

Prove that

$C_1^2 - 2C_2^2 + 3C_3^2 - \ldots\ldots\ldots\ldots\ldots - 2nC_{2n}^2 = (-1)^n n\,C_n.$

Ans. Sol. Given that

$$C_1 + 2C_2x + 3C_3x^2 + \ldots + 2nC_{2n}x^{2n-1} = 2n\,(1+x)^{2n-1} \quad \ldots(1)$$

where $\quad C_r = \dfrac{2n!}{r!(2n-r)!}$

$$[C_1x + C_2x^2 + C_3x^3 + \ldots + C_{2n}x^{2n}]\,{}^x_0 = [(1+x)^{2n}]\,{}^x_0$$

$$\Rightarrow C_1x + C_2x^2 + C_3x^3 + \ldots + C_{2n}x^{2n} = (1+x)^{2n} - 1$$

$$\Rightarrow C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots + C_{2n}x^{2n} = (1+x)^{2n} \quad \ldots(2)$$

Changing x by $-\dfrac{1}{x}$, we get

$$\Rightarrow C_0 - \frac{C_1}{x} + \frac{C_2}{x^2} - \frac{C_3}{x^3} + \ldots + (-1)^{2n}\frac{C_{2n}}{x^{2n}} = \left(1 - \frac{1}{x}\right)^{2n}$$
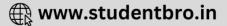
$$\Rightarrow C_0x^{2n} - C_1x^{2n-1} + C_2x^{2n-2} - C_3x^{2n-3} \\ + \ldots + C_{2n} = (x-1)^{2n} \quad \ldots(3)$$

Multiplying eqn. (1) and (3) and equating the coefficients of x^{2n-1} on both sides, we get

$$-C_1^2 + 2C_2^2 - 3C_3^2 + \ldots + 2n\,C_{2n}^2$$

= coeff. of x^{2n-1} in 2n(x - 1) $(x^2 - 1)^{2n-1}$

= 2n [coeff. of x^{2n-2} in $(x^2-1)^{2n-1}$ – coeff. of x^{2n-1} in $(x^2-1)^{2n-1}$]

$$= 2n \left[{}^{2n-1}C_{n-1}(-1)^{n-1} - 0 \right]$$
$$= (-1)^{n-1} \cdot 2n^{2n-1}C_{n-1}$$

$$\Rightarrow C_1^2 - 2C_2^2 + 3C_3^2 + \ldots + 2n\, C_{2n}^2$$

$$= (-1)^n \cdot 2n^{2n-1}C_{n-1} = (-1)^n n \cdot \left(\frac{2n}{n} \cdot {}^{2n-1}C_{n-1} \right)$$

$$= (-1)^n n \cdot {}^{2n}C_n = (-1)^n n \cdot C_n \quad (\because {}^{2n}C_n = C_n)$$

Hence Proved.

Q.2. Prove that $7^{2n} + (2^{3n-3})(3^{n-1})$ is divisible by 25 for any natural number n. (1982 - 5 Marks)

Ans.

Sol. $P(n) : 7^{2n} + 2^{3n-3}, 3^{n-1}$ is divisible by 25 $\forall n \in N$.

Let us prove it by Mathematical Induction :

$P(1) : 7^2 + 2^0 \cdot 3^0 = 49 + 1 = 50$ which is divisible by 25.

$\therefore P(1)$ is true.

Let $P(k)$ be true that is $7^{2k} + 2^{3k-3}, 3^{k-1}$ is divisible by 25.

$\Rightarrow 7^{2k} + 2^{3k-3} \cdot 3^{k-1} = 25m$ where $m \in Z$.

$\Rightarrow 2^{3k-3} \cdot 3^{k-1} = 25m - 7^{2k} \quad \ldots (1)$

Consider $P(k+1) : 7^{2(k+1)} + 2^{3(k+1)-3} \cdot 3^{k+1-1}$

$= 7^{2k} \cdot 7^2 + 2^{3k} \cdot 3^k = 49 \cdot 7^{2k} + 2^3 \cdot 3 \cdot 2^{3k-3} \cdot 3^{k-1}$

$= 49 \cdot 7^{2k} + 24 \, (25m - 7^{2k}) \text{ (Using IH eq. (1))}$

$= 49 \cdot 7^{2k} + 24 \times 25m - 24 \times 7^{2k}$

$= 25 \cdot 7^{2k} + 24 \times 25m = 25 \, (7^{2k} + 24\,m)$

$= 25 \times$ some integral value which is divisible by 25.

∴ P(k + 1) is also true.

Hence by the principle of mathematical induction

P(n) is true ∀ n∈Z .

Q.3. If $(1 + x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n$ then show that the sum of the products of the C_i's taken two at a time, represented by $\underset{0 \le i < j \le n}{\sum \sum} C_i C_j$ is equal to

$2^{2n-1} - \dfrac{(2n)!}{2(n!)^2}$ (1983 - 3 Marks)

Ans.

Sol. $S = \sum \sum C_i C_j$

$0 \le i < j \le n$ **NOTE THIS STEP**

$\Rightarrow S = C_0 (C_1 + C_2 + C_3 + \ldots + C_n) + C_1 (C_2 + C_3 + \ldots + C_n) + C_2 (C_3 + C_4 + C_5 + \ldots C_n)$

$+\ldots C_{n-1}(C_n)$

$\Rightarrow S = C_0 (2^n - C_0) + C_1 (2^n - C_0 - C_1) + C_2(2^n - C_0 - C_1 - C_2) + \ldots + C_{n-1}(2^n - C_0 -$

$C_1 \ldots C_{n-1)} + C_n (2^n - C_0 - C_1 \ldots C_n)$

$\Rightarrow S = 2^n (C_0 + C_1 + C_2 + \ldots + C_{n-1} + C_n)$

$-(C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2) - S$

$\Rightarrow 2S = 2^n.2^n - \dfrac{2n!}{(n!)^2} = 2^{2n} - \dfrac{2n!}{(n!)^2}$

$\Rightarrow S = 2^{2n-1} - \dfrac{2n!}{2(n!)^2}$

Q.4. Use mathematical Induction to prove : If n is an y odd positive integer, then $n(n^2 - 1)$ is divisible by 24. (1983 - 2 Marks)

Ans. Sol. $P(n) : n (n^2 - 1)$ is divisible by 24 for n odd +ve integer.

For n = 2m −1, it can be restated as $P(m) : (2m - 1) (4m^2 - 4m) = 4m (m - 1) (2m - 1)$ is divisible by 24 ∀ m ∈ N

$\Rightarrow$ P(m) : m (m – 1) (2m – 1) is divisible by 6 $\forall$ m $\in$ N .

Here P(1) = 0, divisible by 6.

$\therefore$ P(1) is true.

Let it be true for m = k, i.e.,

k (k–1) (2k–1) = 6p

$\Rightarrow$ 2 k^3 – 3k^2 + k = 6p   ...(1)

Consider P(k + 1) : k (k + 1) (2k + 1) =2k^3 + 3k^2 + k

= 6p + 3k^2 + 3k^2 (Using (1)

= 6 (p + k^2)

$\Rightarrow$ divisible by 6

$\therefore$ P (k + 1) is also true.

Hence P(m) is true $\forall$ m $\in$ N .

Q.5. If p be a natural number then prove that $p^{n+1} + (p + 1)^{2n-1}$ is divisible by $p^2 + p + 1$ for every positive integer n. (1984 - 4 Marks)

Ans. Sol. P(n) : $P^{n+1} + (p + 1)^{2n-1}$ is divisible by $p^2 + p + 1$

For n = 1, P(1) : $p^2 + p + 1$

which is divisible by $p^2 + p + 1$.

$\therefore$ P(1) is true.

Let P(k) be true, i.e., $p^{k+1} + (p + 1)2^{k-1}$ is divisible by $p^2 + p + 1$

$\Rightarrow p^{k+1} + (p + 1)^{2k-1}$

= $(p^2 + p + 1)$ m ....(1)

Consider P(k + 1) : $p^{k+2} + (p + 1)^{2k+1}$

$= p \cdot p^{k+1} + (p + 1)^{2k-1} \cdot (p + 1)^2$

$= p [m (p^2 + p + 1) - (p + 1)^{2k - 1}] + (p + 1)^{2k - 1}(p + 1)^2$

$= p (p^2 + p + 1)m - p (p + 1)^{2k - 1} + (p + 1)^{2k - 1} (p^2 + 2p + 1)$

$= p (p^2 + p + 1)m + (p + 1)^{2k - 1}(p^2 + p + 1)$

$= (p^2 + p + 1) [mp + (p + 1)^{2k - 1}] = (p^2 + p + 1)$
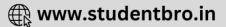
some integral value

$\therefore$ divisible by $p^2 + p + 1$

$\therefore P (k + 1)$ is also true.

Hence by principle of mathematical induction P(n) is true $\forall$ n$\in$N .

Q.6. Given $s_n = 1 + q + q^2 + \ldots\ldots + q^n$;

$S_n = 1 + \dfrac{q+1}{2} + \left(\dfrac{q+1}{2}\right)^2 + \ldots + \left(\dfrac{q+1}{2}\right)^n , q \neq 1$ **Prove that**

$^{n+1}C_1 + {}^{n+1}C_2 s_1 + {}^{n+1}C_3 s_2 + \ldots + {}^{n+1}C_n s_n = 2^n S_n$ **(1984 - 4 Marks)**

Ans. Sol.

We have $s_n = \dfrac{1-q^{n+1}}{1-q}$ ....(1)

and $S_n = \dfrac{1 - \left(\dfrac{q+1}{2}\right)^{n+1}}{1 - \left(\dfrac{q+1}{2}\right)} = \dfrac{2^{n+1} - (q+1)^{n+1}}{2^n(1-q)}$ ....(2)

Now, $^{n+1}C_1 + {}^{n+1}C_2 s_1 + {}^{n+1}C_3 s_2 + \ldots + {}^{n+1}C_{n+1} s_n$

$= \dfrac{1}{1-q}[^{n+1}C_1(1-q) + {}^{n+1}C_2(1-q^2) + {}^{n+1}C_3(1-q^3) + \ldots +$

$+ \ldots + {}^{n+1}C_n(1-q^{n+1})]$ Using (1)

$$= \frac{1}{1-q}\left[\left({}^{n+1}C_1 + {}^{n+1}C_2 + \ldots + {}^{n+1}C_{n+1}\right)\right.$$

$$\left. - \left({}^{n+1}C_1 q + {}^{n+1}C_2 q^2 + \ldots + {}^{n+1}C_{n+1}q^{n+1}\right)\right]$$

$$= \frac{1}{1-q}\left[2^{n+1} - 1\right) - \left\{(1+q)^{n+1} - 1\right\}\right]$$

$$= \frac{2^{n+1} - (1+q)^{n+1}}{(1-q)} = 2^n S_n \quad [\text{Using eq. (2)}]$$

Q.7. Use meth od of mathematical induction $2.7^n + 3.5^n - 5$ is divisible by 24 for all n > 0 (1985 - 5 Marks)

Ans.

Sol. Let $A_n = 2.7^n + 3.5^n - 5$

Then $A_1 = 2.7 + 3.5 - 5 = 14 + 15 - 5 = 24$.

Hence A_1 is divisible by 24.

Now assume that Am is divisible by 24 so that we may write

$A_m = 2.7^m + 3.5^m - 5 = 24k$, $k \in N$ ....(1)

Then $A_{m+1} - A_m$

$= 2(7^{m+1} - 7^m) + 3(5^{m+1} - 5^m) - 5 + 5$

$= 2.7^m(7-1) + 3.5^m(5-1) = 12.(7^m + 5^m)$

Since 7^m and 5^m are odd integers $\forall\ m \in N$, their sum must be an even integer, say $7^m + 5^m = 2p$, $p \in N$.

Hence $A_{m+1} - A_m = 12.2\ p = 24\ p$

or $A_{m+1} = A_m + 24p = 24k + 24p$ [by (1)]

Hence A_{m+1} is divisible by 24.

It follows by mathematical induction that A_n is divisible by 24 for all $n \in N$.

Q.8. Prove by mathematical induction that – (1987 - 3 Marks)

$\dfrac{(2n)!}{2^{2n}(n!)^2} \le \dfrac{1}{(3n+1)^{1/2}}$ for all positive Integers n.

Ans. Sol.

Let $P(n): \dfrac{(2n)!}{2^{2n}(n!)^2} \le \dfrac{1}{(3n+1)^{1/2}}$

For n =1, $P(1): \dfrac{2!}{2^2(1!)^2} \le \dfrac{1}{(3+1)^{1/2}} \Rightarrow \dfrac{1}{4} \le \dfrac{1}{2}$

$\Rightarrow \dfrac{1}{2} \le \dfrac{1}{2}$ which is true for n =1

Assume that P(k) is true, then

$P(k): \dfrac{(2k)!}{2^{2k}(k!)^2} \le \dfrac{1}{(3k+1)^{1/2}}$ ....(1)

For n = k +1,

$\dfrac{[2(k+1)]!}{2^{2(k+1)}[(k+1)!]^2} = \dfrac{(2k+2)!}{2^{2k+2}[(k+1)!]^2}$

$= \dfrac{(2k+2)(2k+1)(2k)!}{4.2^{2k}(k+1)^2(k!)^2}$

$\le \dfrac{(2k+2)(2k+1)}{4(k+1)^2} \cdot \dfrac{1}{(3k+1)^{1/2}}$
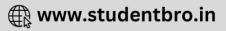
[Using Induction hypothesis (1)]

$= \dfrac{(2k+1)}{2.(k+1)(3k+1)^{1/2}}$

Thus, $\dfrac{[2(k+1)]!}{2^{2(k+1)}[(k+1)!]^2} \le \dfrac{(2k+1)}{2(k+1)(3k+1)^{1/2}}$ ....(2)

In order to prove P(k + 1), it is sufficient to prove that

$$\frac{(2k+1)}{2(k+1)(3k+1)^{1/2}} \leq \frac{1}{(3k+4)^{1/2}} \quad ....(3)$$

Squaring eq. (3), we get

$$\frac{(2k+1)^2}{4(k+1)^2(3k+1)} \leq \frac{1}{3k+4}$$

$$\Rightarrow (2k+1)^2(3k+4) - 4(k+1)^2(3k+1) \leq 0$$

$$\Rightarrow (4k^2 + 4k + 1)(3k+4) - 4(k^2 + 2k + 1)(3k+1) \leq 0$$

$$\Rightarrow (12k^3 + 28k^2 + 19k + 4) - (12k^3 + 28k^2 + 20k + 4) \leq 0$$

$$\Rightarrow -k \leq 0$$

which is true.

Hence from (2) and (3), we get

$$\frac{(2k+2)!}{2^{2k+2}[(k+1)!]^2} \leq \frac{1}{(3k+4)1/2}$$

Hence the above inequation is true for n = k +1 and by the principle of induction it is true for all n ∈ N .

Q.9. Let R = $(5\sqrt{5}+11)^{2n+1}$ and f = R − [R], where [ ] denotes the greatest integer function. Prove that Rf = 4^{2n+4}. (1988 - 5 Marks)

Ans. Sol. We have $5\sqrt{5} - 11 = \frac{4}{5\sqrt{5}+11} < 1$

Therefore $0 < 5\sqrt{5} - 11 < 1$

This gives us $0 < (5\sqrt{5}-11)^{2n+1} < 1$ for every positive integer n.

Also $(5\sqrt{5}+11)^{2n+1} - (5\sqrt{5}-11)^{2n+1}$

$$= 2[^{2n+1}C_1(5\sqrt{5})^{2n}.11 + {}^{2n+1}C_3(5\sqrt{5})^{2n-2}.11^3 +$$

$$.... + {}^{2n+1}C_{2n+1}11^{2n+1}]$$

$$= 2[^{2n+1}C_1(125)^n.11 + {}^{2n+1}C_3(125)^{n-1}.11^3 +$$

$$.... + {}^{2n+1}C_{2n+1}11^{2n+1}]$$

$$= 2k$$

....(1)

where k is some positive integer.

Let F $= (5\sqrt{5} - 11)^{2n+1}$

Then equation (1) becomes R − F = 2k

$\Rightarrow$ [R] + R − [R]− F = 2k $\Rightarrow$ [R] + f − F = 2k

$\Rightarrow$ f − F = 2k − [R] $\Rightarrow$ f − F is an integer..

But $0 \leq f < 1$ and $0 < F < 1$

Therefore $-1 < f - F < 1$

Since f − F is an integer, we must have f − F = 0

$\Rightarrow$ f = F.

Now, Rf = RF $=: (5\sqrt{5} + 11)^{2n+1}(5\sqrt{5} - 11)^{2n+1}$

$$= [(5\sqrt{5})^2 - 12]^{2n+1} = 4^{2n+1}$$

Q.10. Using mathematical induction, prove that(1989 - 3 Marks)
$^mC_0\,{}^nC_k + {}^mC_1\,{}^nC_{k-1} + \text{...............} {}^mC_k\,{}^nC_0 = {}^{(m+n)}C_k,$ **where m, n, k are positive integers,**
and $^pC_q = 0$ for p < q.

Ans.

Sol. Let the given statement be

$P(m,n): {}^mC_0{}^nC_k + {}^mC_1{}^nC_{k-1} + \ldots + {}^mC_k{}^nC_0 = {}^{m+n}C_k$

where m, n, k $\in$ N and ${}^PC_q = 0$ for p<q.

As k is a positive integer and ${}^P C_q = 0$ for p<q .

$\therefore$ k must be a positive integer less than or equal to the smaller of m and n,

We have k = 1, when m = n = 1

$\therefore \quad P(1,1)$ is ${}^1C_0\,{}^1C_1 + {}^1C_1\,{}^1C_0 = {}^2C_1 \Rightarrow 1 + 1 = 2.$

Thus P (1, 1) is true.

Now let us assume that P(m, n) holds good for any fixed value of m and n i.e.

$${}^mC_0{}^nC_k + {}^mC_1{}^nC_{k-1} + \ldots + {}^mC_k{}^nC_0 = {}^{m+n}C_k \quad \ldots (1)$$

$${}^{m+1}C_0{}^{n+1}C_k + {}^{m+1}C_1{}^{n+1}C_{k-1} + \ldots + {}^{m+1}C_k{}^{n+1}C_0$$

$$= {}^{m+n+2}C_k \quad \ldots (2)$$

Consider LHS

$= {}^{m+1}C_0{}^{n+1}C_k + {}^{m+1}C_1{}^{n+1}C_{k-1} + \ldots + {}^{m+1}C_k{}^{n+1}C_0$

$= 1.({}^nC_{k-1} + {}^nC_k) + ({}^mC_0 + {}^mC_1)({}^nC_{k-2} + {}^nC_{k-1})$

$+ ({}^mC_1 + {}^mC_2)({}^nC_{k-3} + {}^nC_{k-2}) + \ldots + ({}^mC_{k-1} + {}^mC_k).1$

$= ({}^nC_{k-1} + {}^mC_1{}^nC_{k-2} + {}^mC_2{}^nC_{k-3} + \ldots + {}^mC_{k-1}{}^nC_0)$

$+ ({}^nC_k + {}^mC_1{}^nC_{k-1} + {}^mC_2{}^nC_{k-2} + \ldots + {}^mC_{k-1}{}^nC_1 + {}^mC_k)$

$+ ({}^mC_0{}^nC_{k-2} + {}^mC_1{}^nC_{k-3} + \ldots + {}^mC_{k-2}{}^nC_0)$

$+ ({}^mC_0{}^nC_{k-1} + {}^mC_1{}^nC_{k-2} + {}^mC_2{}^nC_{k-3}$

$+ \ldots + {}^mC_{k-2}{}^nC_1 + {}^mC_{k-1})$

$= {}^{m+n}C_{k-1} + {}^{m+n}C_k + {}^{m+n}C_{k-2} + {}^{m+n}C_{k-1}$ [Using (1)]

$= {}^{m+n+1}C_k + {}^{m+n+1}C_{k-1} = {}^{m+n+2}C_k$

Hence the theorem holds for the next integers $m + 1$ and $n + 1$. Then by mathematical induction the statement P (m, n) holds for all positive integral values of m and n.

Q.11. Prove that

$C_0 - 2^2C_1 + 3^2C_2 - .... + (-1)^n (n+1)^2C_n = 0$, n > 2, where $C_r = {}^nC_r$.

Ans. Sol. We know that $(1 - x)^n = C_0 - C_1x + C_2x^2 - C_3x^3 + ... + (-1)^n C_nx^n$

Multiplying both sides by x, we get

$x(1 - x)^n = C_0 x - C_1x^2 + C_2x^3 - C_3x^4 + ... + (-1)^n C_nx^{n+1}$

Differentiating both sides w.r. to x, we get

$(1 - x)^n - nx (1 - x)^{n} - 1 = C_0 - 2C_1 x + 3C_2 x^2 - 4C_3x^3 + ... + (-1)^n (n + 1) C_nx^n$

Again multiplying both sides by x, we get

$x (1 - x)^n - nx^2 (1 - x)n - 1 = C_0x - 2C_1x^2 + 3C_2x^3 - 4C_3x^4 + ... + (-1)^n (n + 1) C_nx^{n+1}$

Differentiating above with respect to x, we get

$(1 - x)^n - nx (1- x)^{n-1} - 2nx (1- x)^{n-1} + nx^2 (n - 1) (1 - x)^{n-2}$

$= C_0 - 2^2 C_1 x + 32 C_2 x^2 - 4^2 C_3 x^3 + ... + (-1)^n (n + 1)^2 C_n x^n$

Substituting x = 1, in above, we get

$0 = C_0 - 2^2C_1 + 3^2C_2 - 4^2C_3 + ... + (-1)^n (n + 1)^2C_n$

Hence Proved.

Q.12. Prove that $\dfrac{n^7}{7} + \dfrac{n^5}{5} + \dfrac{2n^3}{3} - \dfrac{n}{105}$ **is an integer for every positive integer n. (1990 - 2 Marks)**

Ans. Sol. We have

$P(n)$: $\dfrac{n^7}{7} + \dfrac{n^5}{5} + \dfrac{2n^3}{3} - \dfrac{n}{105}$ is an integer, $\forall \; n \in N$

$P(1)$: $\dfrac{1}{7} + \dfrac{1}{5} + \dfrac{2}{2} - \dfrac{1}{105}$

$= \dfrac{15 + 21 + 70 - 1}{105} = \dfrac{105}{105} = 1$ an integer

$\therefore P(1)$ is true

Let $P(k)$ be true i.e.

$\dfrac{k^7}{7} + \dfrac{k^5}{5} + \dfrac{2k^3}{3} - \dfrac{k}{105}$ is an integer

$\Rightarrow \quad \dfrac{k^7}{7} + \dfrac{k^5}{5} + \dfrac{2k^3}{3} - \dfrac{k}{105} = m$, (say)

$m \in N$ ....(1)

Consider $P(k + 1)$:

$= \dfrac{(k+1)^7}{7} + \dfrac{(k+1)^5}{5} + \dfrac{2(k+1)^3}{3} - \dfrac{(k+1)}{105}$

$= \left(\dfrac{k^7 + 7k^6 + 21k^5 + 35k^4 + 35k^3 + 21k^2 + 7k + 1}{7} \right)$

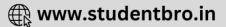$+ \left(\dfrac{k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1}{5} \right)$

$+ 2 \left(\dfrac{k^3 + 3k^2 + 3k + 1}{3} \right) - \left(\dfrac{k+1}{105} \right)$

$= \left(\dfrac{k^7}{7} + \dfrac{k^5}{5} + \dfrac{2k^3}{3} - \dfrac{k}{105} \right)$

$+ [k^6 + 3k^5 + 5k^4 + 5k^3 + 3k^2 + k + k^4$

$+ 2k^3 + 2k^2 + k + 2k^2 + 2k] + \left(\dfrac{1}{7} + \dfrac{1}{5} + \dfrac{2}{3} - \dfrac{1}{105} \right)$

$= m +$ some integral value $+ 1$

= some integral value

∴ P (k + 1) is also true.

Hence P (n) is true ∀ n ∈N , (by the Principle of Mathematical Induction.)

Q.13. Using in duction or otherwise, prove that for any nonnegative integers m, n, r and k, (1991 - 4 Marks)

$$\sum_{m=0}^{k} (n-m)\frac{(r+m)!}{m!} = \frac{(r+k+1)!}{k!}\left[\frac{n}{r+1} - \frac{k}{r+2}\right]$$

Ans. Sol. Let $P(k) = \sum_{m=0}^{k} \frac{(n-m)(r+m)!}{m!} = \frac{(r+k+1)!}{k!}\left[\frac{n}{r+1} - \frac{k}{r+2}\right]$

For k = 1, we will have two terms, on LHS, in sigma for m = 0 and m = 1, so that

$$\text{LHS} = (n-0)\frac{r!}{0!} + (n-1)\frac{(r+1)!}{1!}$$

and $\text{RSH} = \frac{(r+2)!}{1!}\left[\frac{n}{r+1} - \frac{1}{r+2}\right]$

Hence LHS = RHS for k = 1.

Now let the formula holds for k = s, that is let

$$\sum_{m=0}^{s} \frac{(n-m)(r+m)!}{m!} = \frac{(r+s+1)!}{s!}\left(\frac{n}{r+1} - \frac{s}{r+2}\right) ...(1)$$

Let us add next term corresponding to m = s + 1 i.e.

adding $\frac{(n-s-1)(r+s+1)!}{(s+1)!}$ to both sides, we get

$$\sum_{m=0}^{s+1} \frac{(n-m)(r+m)!}{m!} = \frac{(r+s+1)!}{s!}\left[\frac{n}{r+1} - \frac{s}{r+2}\right]$$

$$+ \frac{(n-s-1)(r+s+1)!}{(s+1)!}$$

$$= \frac{(r+s+1)!}{(s+1)!}\left[\frac{(s+1)n}{r+1} - \frac{s(s+1)}{r+2} + n-s-1\right]$$

$$= \frac{(r+s+1)!}{(s+1)!}\left[n\left\{\frac{s+1}{r+1}+1\right\}-(s+1)\left\{\frac{s}{r+2}+1\right\}\right]$$

$$= \frac{(r+s+2)(r+s+1)!}{(s+1)!}\left[\frac{n}{r+1}-\frac{s+1}{r+2}\right]$$

Hence the formula holds for k = s + 1 and so by the induction principle, the formula holds for all natural numbers k.

Q.14. If $\displaystyle\sum_{r=0}^{2n} a_r(x-2)^r = \sum_{r=0}^{2n} b_r(x-3)^r$ **and ak = 1 for all k ≥ n, then show that** $b_n = {}^{2n+1}C_{n+1}$ **(1992 - 6 Marks)**

Ans.

Sol. Given that

$$\sum_{r=0}^{2n} a_r(x-2)^r = \sum_{r=0}^{2n} b_r(x-3)^r \quad ....(1)$$

and $a_k = 1, \forall k \geq n$

To prove $b_n = {}^{2n+1}C_{n+1}$ In the given equation (1) let us put x −3 = y so that x − 2 = y + 1 and we get

$$\sum_{r=0}^{2n} a_r(1+y)^r = \sum_{r=0}^{2n} b_r(y)^r$$

$$\Rightarrow a_0 + a_1(1+y) + \ldots a_{n-1} + (1+y)^{n-1}(1+y)^n$$
$$+ (1+y)^{n+1} + \ldots (1+y)^{2n}$$

$$= \sum_{r=0}^{2n} b_r y^r \text{ [Using } a_k = 1, \forall k \geq n]$$

NOTE THIS STEP :

$$\Rightarrow {}^nC_n + {}^{n+1}C_n + {}^{n+2}C_n + \ldots + {}^{2n}C_n = b_n$$

$$\Rightarrow ({}^{n+1}C_{n+1} + {}^{n+1}C_n) + {}^{n+2}C_n + \ldots + {}^{2n}C_n = b_n$$

[Using ${}^nC_n = {}^{n+1}C_{n+1} = 1$]

$$\Rightarrow b_n = {}^{n+2}C_{n+1} + {}^{n+2}C_n + \ldots + {}^{2n}C_n$$

[Using ${}^mC_r + {}^mC_{r-1} = {}^{m+1}C_r$]

Combining the terms in similar way, we get

$$\Rightarrow b_n = {}^{2n}C_{n+1} + {}^{2n}C_n \Rightarrow b_n = {}^{2n+1}C_{n+1}$$

Hence Proved

Q.15. Let p ≥ 3 be an integer and α, β be the roots of $x^2 - (p+1)x + 1 = 0$ using mathematical induction show that $\alpha^n + \beta^n$.
(i) is an integer and (ii) is not divisible by p (1992 - 6 Marks)

Ans.

Sol. Since α, β are the roots of $x^2 - (p+1)x + 1 = 0$
∴α + β = p + 1; αβ = 1
Here p ≥ 3 and p ∈ Z

(i) To prove that $\alpha^n + \beta^n$ is an integer.

Let us consider the statement, "$\alpha^n + \beta^n$ is an integer."

Then for n = 1, α + β = p + 1 which is an integer, p being an integer.

∴Statement is true for n = 1

Let the statement be true for n ≤ k, i.e., $\alpha^k + \beta^k$ is an integer Then ,

$$\alpha^{k+1} + \beta^{k+1} = \alpha^k \cdot \alpha + \beta^k \cdot \beta$$

$$= \alpha(\alpha^k + \beta^k) + \beta(\alpha^k + \beta^k) - \alpha\beta^k - \alpha^k\beta$$

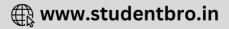$$= (\alpha + \beta)(\alpha^k + \beta^k) - \alpha\beta(\alpha^{k-1} + \beta^{k-1})$$

$$= (\alpha + \beta)(\alpha^k + \beta^k) - (\alpha^{k-1} + \beta^{k-1}) \ldots(1) \text{ [as } \alpha\beta = 1]$$

= difference of two integers = some integral value

⇒ Statement is true for n = k + 1.

∴By the principle of mathematical induction the given statement is true for ∀ n∈ N .

(ii) Let R_n be the remainder of $\alpha^n + \beta^n$ when divided by p where $0 \le R_n \le p-1$

Since $\alpha + \beta = p + 1$ $\quad \therefore R_1$

$= 1$ Also $\alpha^2 + \beta^2 = (\alpha + b)^2 - 2\alpha\beta = (p + 1)^2 - 2$

$= p^2 + 2p - 1 = p(p + 1) + p - 1$

$\therefore R_2 = p - 1$

Also from equation (1) of previous part

(i), we have $\alpha^{n+1} + \beta^{n+1} = (p + 1)(\alpha^n + \beta^n) - (\alpha^{n-1} + \beta^{n-1})$ $\quad =$

$p(\alpha^n + \beta^n) + (\alpha n + \beta n) - (\alpha^{n-1} + \beta^{n-1})$

$\Rightarrow R_{n+1}$ is the remainder of $R_n - R_{n-1}$ when divided by p

$\therefore$ We observe that $R_2 - R_1 = p - 1 - 1$

$\therefore R_3 = p - 2$

Similarly, R_4 is the remainder when $R_3 - R_2$ is divided by p

where $R_3 - R_2 = p - 2 - p + 1 = -1 = -p + (p - 1)$ $\quad \therefore R_4 = p - 1$

$R_4 - R_3 = p - 1 - p + 1 = 1$ $\qquad \therefore R_5 = 1$

$R_5 - R_4 = 1 - p + 1 = -p + 2$ $\qquad \therefore R_6 = p - 2$

It is evident for above that the remainder is either 1 or $p - 1$ or $p - 2$.

Since $p \ge 3$, so none is divisible by p.

Q.16. Using mathematical induction, prove that $\tan^{-1}(1/3) + \tan^{-1}(1/7) + ... \tan^{-1}\{1/(n^2+n+1)\} = \tan^{-1}\{n/(n+2)\}$ (1993 - 5 Marks)

Ans. Sol. To prove

$P(n): \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) + ...\tan^{-1}\left(\frac{1}{n^2+n+1}\right)$

$= \tan^{-1}\left(\frac{n}{n+2}\right)$

For n = 1, LHS $= \tan^{-1}\frac{1}{3}$;

RHS $= \tan^{-1}\frac{1}{3}$ ⇒ LHS = RHS.

∴P(1) is true.

Let P(k) be true, i.e.

$$\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \ldots \ldots \tan^{-1}\left(\frac{1}{k^2+k+1}\right) = \tan^{-1}\left(\frac{k}{k+2}\right)$$

Consider P (k + 1)

$$\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} + \ldots \tan^{-1}\left(\frac{1}{k^2+k+1}\right)$$

$$+ \tan^{-1}\left(\frac{1}{(k+1)^2+(k+1)+1}\right)$$

$$= \tan^{-1}\left[\frac{k+1}{(k+1)+2}\right]$$

LHS $= \tan^{-1}\left[\frac{k}{(k+2)}\right] + \tan^{-1}\left(\frac{1}{(k^2+3k+3)}\right)$ [Using equation (1)]

$$= \tan^{-1}\left[\frac{\frac{k}{(k+2)} + \frac{1}{k^2+3k+3}}{1 - \left(\frac{k}{k+2}\right)\left(\frac{1}{k^2+3k+3}\right)}\right]$$

$$= \tan^{-1}\left[\frac{(k+1)(k^2+2k+2)}{(k+3)(k^2+2k+2)}\right] = \tan^{-1}\left(\frac{k+1}{k+3}\right) = RHS$$

∴P(k + 1) is also true.

Hence by the principle of mathematical induction P(n) is true for every natural number.

Q. 17. Prove that $\displaystyle\sum_{r=1}^{k}(-3)^{r-1}\,{}^{3n}C_{2r-1} = 0$ **= 0, where k = (3n) /2 and n is an even positive integer. (1993 - 5 Marks)**

Ans. Sol.

To evaluate $\sum_{r=1}^{k}(-3)^{r-1}\,^{3n}C_{2r-1}$ where $k=\dfrac{3n}{2}$.

and n is +ve even interger.

Let n = 2m, where $m \in z^{+}$ $\therefore$ $k=\dfrac{3(2m)}{2}=3m$

$\therefore$ $\displaystyle\sum_{r=1}^{k}(-3)^{r-1}\,^{3n}C_{2r-1}=\sum_{r=1}^{3m}(-3)^{r-1}\,^{6m}C_{2r-1}$

$=\,^{6m}C_{1}-3.\,^{6m}C_{3}+3^{2}\,^{6m}C_{5}-\cdots$ ...(1)

Now we know that

$(1+a)^{6m}-(1-a)^{6m}$
$=2[^{6m}C_{1}a+\,^{6m}C_{3}a^{3}+\,^{6m}C_{5}a^{5}+\ldots]$ ...(2)

Keeping in mind the form of RHS in equation (1) and in equation (2)

We put $a=i\sqrt{3}$ in equation (2) to get

$(1+i\sqrt{3})^{6m}-(1+i\sqrt{3})^{6m}$

$=2[^{6m}C_{1}i\sqrt{3}-\,^{6m}C_{3}i3\sqrt{3}+\,^{6m}C_{5}i3^{2}\sqrt{3}\ldots]$

$\Rightarrow (1+i\sqrt{3})^{6m}-(1-i\sqrt{3})^{6m}$

$=2\sqrt{3}i[^{6m}C_{1}-3.\,^{6m}C_{3}+3^{2}\,^{6m}C_{5}\ldots]\ldots(3)$

But $1+i\sqrt{3}=2(\cos\pi/3+i\sin\pi/3)$

$\therefore (1+i\sqrt{3})^{6m}=2^{6m}(\cos\pi/3+i\sin\pi/3)^{6m}$

NOTE THIS STEP

$=2^{6m}\left(\cos\dfrac{6m\pi}{3}+i\sin\dfrac{6m\pi}{3}\right)$ [Using D' Moivre's thm.]

Similarly,

$$(1-i\sqrt{3})^{6m} = 2^{6m}\left(\cos\frac{6m\pi}{3} - i\sin\frac{6m\pi}{3}\right)$$

$$\therefore \quad (1+i\sqrt{3})^{6m} - (1-i\sqrt{3})^{6m} = 2^{6m}.2\sin 2m\pi = 0$$

Substituting the above in equation (3) we get

$$^{6m}C_1 - 3.^{6m}C_3 + 3^2\ ^{6m}C_5 - ..... = 0$$

$$\Rightarrow \quad \sum_{r=1}^{k}(-3)^{r-1}\ ^{3n}C_{2r-1} = 0.$$

Hence Proved

Q.18. If x is not an integral multiple of 2π use mathematical induction to prove that : (1994 - 4 Marks)

$$\cos x + \cos 2x + ... + \cos nx = \cos\frac{n+1}{2}x\sin\frac{nx}{2}\operatorname{cosec}\frac{x}{2}$$

Ans. Sol. Let $P(n): \cos x + \cos 2x + ...+ \cos nx$

$$= \cos\frac{n+1}{2}x\sin\frac{nx}{2}\cos ec\frac{x}{2}\ ....(1)$$

where x is not an integral multiple of 2 p .

For n = 1 P (1) : L.H.S. $= \cos x$

$$\text{R.H.S.} = \cos\frac{1+1}{2}x\sin\frac{x}{2}\cos ec\frac{x}{2} = \cos x$$

L.H.S. = R.H.S.

$\Rightarrow$ P (1) is true.

Let P(k) be true i.e.

$\cos x + \cos 2x + ...+ \cos kx$

$$= \cos\frac{k+1}{2}x\sin\frac{kx}{2}\operatorname{cosec}\frac{x}{2}\ ....(2)$$

Consider P(k + 1) :

cos x + cos 2x + ...+ cos kx + cos (k + 1) x

$$= \cos\left(\frac{k+2}{2}\right)x \sin\frac{(k+1)x}{2}\cosec\frac{x}{2}$$

.L.H.S. [cos x + cos 2x + ...+ cos kx + cos (k + 1) x

$$= \cos\left(\frac{k+1}{2}\right)x\sin\cosec\frac{kx}{2}\ \frac{x}{2}+\cos(k+1)x\ [\text{Using (2)}]$$

$$= \left[\cos\left(\frac{k+1}{2}\right)x\sin\frac{kx}{2}+\cos(k+1)x\sin\frac{x}{2}\right]\cosec\frac{x}{2}$$

$$= \frac{1}{2}\left[2\cos\frac{(k+1)x}{2}\sin\frac{kx}{2}+2\cos(k+1)x\sin\frac{x}{2}\right]\cosec\frac{x}{2}$$

$$= \frac{1}{2}[\sin\left(\frac{2k+1}{2}\right)x-\sin\frac{x}{2}$$

$$+\sin\left(xk+\frac{3x}{2}\right)-\sin\left(xk+\frac{x}{2}\right)]\cosec\frac{x}{2}$$

$$= \frac{1}{2}\left[\sin\left(xk+\frac{3x}{2}\right)-\sin\frac{x}{2}\right]\cosec\frac{x}{2}$$

$$= \frac{1}{2}\left[2\cos\frac{(k+2)x}{2}\sin\frac{(k+1)}{2}\right]\cosec\frac{x}{2}\ =\ \text{R.H.S.}$$

∴P (k + 1) is also true.

Hence by the principle of mathematical induction

P (n) is true ∀ n∈ N .

Q.19. Let n be a positive integer and (1994 - 5 Marks)

$(1 + x + x^2)^n = a_0 + a_1x + ...+ a_{2n}\, x^{2n}$

Show that $a_0^2 - a_1^2 + a_2^2 ....+ a_{2n}^2 = a_n$

Ans. Sol. Given that,

$(1 + x + x^2)^n = a_0 + a_1x +....+ a_{2n}x^{2n}$ ....(1)

where n is a +ve integer.

Replacing x by $-\frac{1}{x}$ in eq n (1), we get

$$\left(1-\frac{1}{x}+\frac{1}{x^2}\right)^n = a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} - \frac{a_3}{x^3} + \dots + \frac{a_{2n}}{x^{2n}} \quad \dots(2)$$

Multiplying eq.'s (1) and (2) :

$$\frac{(1+x+x^2)^n (x^2-x+1)^n}{x^{2n}}$$

$$= (a_0 + a_1 x + \dots + a_{2n}x^{2n})\left(a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} + \dots + \frac{a_n}{x^{2n}}\right)$$

Equating the constant terms on both sides we get

$$a_0^2 - a_1^2 + a_2^2 - a_3^2 + \dots + a_{2n}^2 = \text{constant term in the expansion of}$$

$$\frac{[(1+x+x^2)(1-x+x^2)]^n}{x^{2n}}$$

= Coeff. of x^{2n} in the expansion of $(1 + x^2 + x^4)^n$ But replacing x by x^2 in eq's (1), we have

$(1 + x^2 + x^4)^n = a_0 + a_1 x^2 + \dots + a_{2n} (x^2)^{2n}$

∴Coeff of $x^{2n} = a_n$

Hence we obtain, $a_0^2 - a_1^2 + a_2^2 - a_3^2 + \dots + a_{2n}^2 = a_n$

Q.20. Using mathematical induction prove that for every integer n ≥ 1, $(3^{2n}-1)$ is divisible by 2^{n+2} but not by 2^{n+3}. (1996 - 3 Marks)

Ans. Sol. For n = 1, $3^{2n} - 1 = 3^{21} - 1 = 9 - 1 = 8$ which is divisible by $2^{n+2} = 2^3 = 8$ but is not divisible by $2^{n+3} = 2^4 = 16$

Therefore, the result is true for n = 1.

Assume that the result is true for n = k.

That is, assume that $3^{2k} - 1$ is divisible by 2^{k+2} but is not divisible by 2^{k+3},

Since $3^{2k} - 1$ is divisble by 2^{k+2} but not by 2^{k+3},

we can write $3^{2^k} - 1 = (m) 2^{k+2}$ where m must be an odd positive integer, for otherwise $3^{2k} - 1$ will become divisible by 2^{k+3}.

For n = k +1, we have $3^{2^{k+1}} - 1 = 3^{2^k \cdot 2} - 1 = \left(3^{2^k}\right)^2 - 1$

$= (m.2^{k+2} + 1)^2 - 1$ [Using (1)]

$= m^2.(2^{k+2})^2 + 2m.2^{k+2} + 1 - 1$

$= m^2.2^{2k+4} + m.2^{k+3}$

$= 2^{k+3}(m^2.2^{k+1} + m.)$

$\Rightarrow 3^{2k+1} - 1$ is divisible by 2^{k+3}.

But $3^{2k+1} - 1$ is not divisible by 2^{k+4} for otherwise we must have 2 divides $m^2. 2^{k+1} + m$.

But this is not possible as m is odd.

Thus, the result is true for n = k + 1.

Q.21. Let $0 < A_i < p$ for i = 1, 2 ...., n. Use mathematical induction to prove that

$\sin A_1 + \sin A_2 ... + \sin A_n \leq n \sin \left(\dfrac{A_1 + A_2 + ... + A_n}{n} \right)$

where ≥ 1 is a natural number. {You may use the fact that p sin x + (1−p) sin y ≤ sin [px + (1−p)y], where $0 \leq p \leq 1$ and $0 \leq x, y \leq \pi$} (1997 - 5 Marks)

Ans.

Sol. For n = 1, the inequalitity becomes $\sin A_1 \leq \sin A_1$, which is clearly true.

Assume that the inequality holds for n = k where k is some positive integer. That is, assume that

$\sin A_1 + \sin A_2 + ... + \sin A_k \leq k \sin \leq k \sin \left(\dfrac{A_1 + A_2 + ... + A_k}{k} \right)$ ....(1)

for same positive integer k.

We shall now show that the result holds for n = k + 1 that is, we show that

$\sin A_1 + \sin A_2 + \dots + \sin A_k + \sin A_{k+1}$

$$\leq (k+1)\sin\left(\frac{A_1 + A_2 + \dots + A_{k+1}}{k+1}\right) \dots (2)$$

L.H.S. of (2) = $\sin A_1 + \sin A_2 + \dots + \sin A_k + \sin A_{k+1}$

$$\leq k\sin\left(\frac{A_1 + A_2 + \dots + A_k}{k}\right) + \sin A_{k+1}$$

[Induction assumption]

$$= (k+1)\left[\frac{k}{k+1}\sin\alpha + \frac{1}{k+1}\sin A_{k+1}\right];$$

where $\quad \alpha + \dfrac{A_1 + A_2 + \dots A_k}{k}$

$$\therefore \text{L.H.S. of (2)} \leq (k+1)\left[\left(1 - \frac{k}{k+1}\right)\sin\alpha + \frac{1}{k+1}\sin A_{k+1}\right]$$

$$\leq (k+1)\sin\left\{\left(1 - \frac{k}{k+1}\right)\alpha + \frac{1}{k+1}A_{k+1}\right\}$$

[Using the fact p sin x + (1– p) sin y ≤ sin [px + (1– p)y]

$0 \leq p \leq 1, 0 \leq x, y \leq \pi$]

$$\left\{\frac{k}{k+1}\left(\frac{A_1 + A_2 + \dots A_k}{k}\right) + \frac{1}{k+1}A_{k+1}\right\}$$

$$\left(\frac{A_1 + A_2 + \dots A_{k+1}}{k+1}\right)$$

Thus, the inequality holds for n = k + 1. Hence, by the principle of mathematical induction the inequality holds for all n∈ N.

Q.22. Let p be a prime and m a positive integer. By mathematical induction on m, or otherwise, prove that whenever r is an integer such that p does not divide r, p divides $^{mp}C_r$, (1998 - 8 Marks)

[Hint: You may use the fact that $(1+x)^{(m+1)p} = (1 + x)^p (1 + x)^{mp}$]

Ans. Sol. We know that $^nC_r = \dfrac{n}{r} \, ^{n-1}C_{r-1}$

$\therefore \quad ^{mp}C_r = \dfrac{mp}{r} \, ^{mp-1}C_{r-1}$

$= \left[\dfrac{m \cdot \, ^{mp-1}C_{r-1}}{r}\right] p$

Now, L.H.S is an integer

$\Rightarrow$ RHS must be an integer

But p and r are coprime (given)

$\therefore$ r must divide m. $^{mp-1}C_{r-1}$

or $\dfrac{m \cdot \, ^{mp-1}C_{r-1}}{r}$ is an integer..

$\Rightarrow \dfrac{^{mp}C_r}{p}$ is an integer or $^{mp}C_r$ is divisible by p.
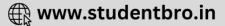
Q.23. Let n be any positive integer. Prove that (1999 - 10 Marks)

$$\sum_{k=0}^{m} \frac{\binom{2n-k}{k}}{\binom{2n-k}{n}} \cdot \frac{(2n-4k+1)}{(2n-2k+1)} 2^{n-2k} = \frac{\binom{n}{m}}{\binom{2n-2m}{n-m}} 2^{n-2m}$$

for each non-be gatuve integer $m \le n$. $m \le n$. $\left(\text{Here} \binom{p}{q} = \, ^pC_q\right)$.

Ans. Sol.

Let $P(m) = \displaystyle\sum_{k=0}^{m} \frac{\binom{2n-k}{k}^{(2n-4k+1)}}{\binom{2n-k}{n}^{(2n-2k+1)}} 2^{n-2k}$

$= \dfrac{\binom{n}{m}}{\binom{2n-2m}{n-m}} \cdot 2^{n-2m}$ ....(1)

For m = 0, LHS $= \dfrac{\dbinom{2n}{0}}{\dbinom{2n}{n}} \cdot \dfrac{2n+1}{2n+1} \cdot 2^n = \dfrac{1}{\dbinom{2n}{n}} 2^n,$

R.H.S. $= \dfrac{\dbinom{n}{0}}{\dbinom{2n}{n}} \cdot 2^n = \dfrac{1}{\dbinom{2n}{n}} 2^n =$ LHS

$[\because m = 0 \Rightarrow k = 0]$

$\therefore$ P(0) holds true. Now assuming P (m)

L.H.S. of P(m + 1) = L.H.S. of

$$P(m) + \dfrac{\dbinom{2n-m-1}{m+1}}{\dbinom{2n-m-1}{n}} \cdot \dfrac{(2n-4m-3)}{(2n-2m-1)} \cdot 2^{n-2m-2}$$

$$= \dfrac{n!(n-m)!}{m!(2n-2m)!} \cdot 2^{n-2m}$$

$$+ \dfrac{n!(n-m-1)!(2n-4m-3)}{(m+1)!(2n-2m-2)!(2n-2m-1)} \cdot 2^{n-2m-2}$$

$$= \dfrac{n!(n-m-1)!2^{n-2m-2}}{(m+1)!(2n-2m-1)!}$$

$$\times \left\{ \dfrac{(n-m).4\,(m+1)}{(2n-2m)} + (2n-4m-3) \right\}$$

$$= \dfrac{n!(n-m-1)!2^{n-2m-2}(2n-2m-1)}{(m+1)!(2n-2m-1)!}$$

$$= \dfrac{n!(n-m-1)!2^{n-2m-2}}{(m+1)!(2n-2m-2)!} = \dfrac{\dbinom{n}{m+1}}{\dbinom{2n-2m-2}{n-m-1}} \cdot 2^{n-2m-2}$$

= R.H.S. of P(m + 1).

Hence by mathematical induction, result follows for all $0 \le m \le n$.

Q.24. For any positive integer m, n (with n ≥ m), let $\binom{n}{m} = {}^{n}C_m$.

Prove that $\binom{n}{m} + \binom{n-1}{m} + \binom{n-2}{m} + \cdots + \binom{m}{m} = \binom{n+1}{m+2}$

Hence or otherwise, prove that

$\binom{n}{m} + 2\binom{n-1}{m} + 3\binom{n-2}{m} + \cdots + (n-m+1)\binom{m}{m} = \binom{n+2}{m+2}$.   **(2000 - 6 Marks)**

Ans. Sol. Given that for positive integers m and n such that n ≥ m, then to prove that

$${}^{n}C_m + {}^{n-1}C_m + {}^{n-2}C_m + \cdots + {}^{m}C_m = {}^{n+1}C_{m+1}$$

L.H.S. ${}^{m}C_m + {}^{m+1}C_m + {}^{m+2}C_m + \cdots + {}^{n-1}C_m + {}^{n}C_m$

[writing L.H.S. in reverse order]

$= ({}^{m+1}C_{m+1} + {}^{m+1}C_m) + {}^{m+2}C_m + \cdots + {}^{n-1}C_m + {}^{n}C_m$

$[\because {}^{m}C_m = {}^{m+1}C_{m+1}]$

$= ({}^{m+2}C_{m+1} + {}^{m+2}C_m) + {}^{m+3}C_m + \cdots + {}^{n}C_m$

$[\because {}^{n}C_{r+1} + {}^{n}C_r = {}^{n+1}C_{r+1}]$

$= {}^{m+3}C_{m+1} + {}^{m+3}C_m + \cdots + {}^{n}C_m$

Combining in the same way we get

$= {}^{n}C_{m+1} + {}^{n}C_m = {}^{n+1}C_{m+1} = $ R.H.S.

Again we have to prove ${}^{n}C_{m+2} \ {}^{n-1}C_{m+3} \ {}^{n-2}C_m + \cdots + (n-m+1)\,{}^{m}C_m = {}^{n+2}C_{m+2}$

$= [{}^{n}C_m + {}^{n-1}C_m + {}^{n-2}C_m + \cdots + {}^{m}C_m] + [{}^{n-1}C_m + {}^{n-2}C_m + \cdots + {}^{m}C_m] + [{}^{n-2}C_m + \cdots + {}^{m}C_m] + \cdots + [{}^{m}C_m]$

[n – m + 1 bracketed terms] $= {}^{n+1}C_{m+1} + {}^{n}C_{m+1} \ {}^{n-1}C_{m+1} \cdots + {}^{m+1}C_{m+1}$

[using previous result.]

$= {}^{n+2}C_{m+2}$

[Replacing n by n + 1 and m by m + 1 in the previous result.] = R.H.S.

Q.25. For every positive integer n, prove that $\sqrt{(4n+1)} < \sqrt{n} + \sqrt{n+1} < \sqrt{4n+2}$ **Hence or other wise,**

prove that $[\sqrt{n} + \sqrt{(n+1)}] = [\sqrt{4n+1}]$ **where [x] denotes the greatest integer not exceeding x. (2000 - 6 Marks)**

Ans. Sol. For n > 0 $\sqrt{4n+1} > 0, \sqrt{n} + \sqrt{n+1} > 0$ and $\sqrt{4n+2} > 0$

Now, $\sqrt{4n+1} < \sqrt{n} + \sqrt{n+1} < \sqrt{4n+2}$ to be proved.

I. To prove $\sqrt{4n+1} < \sqrt{n} + \sqrt{n+1}$

Squaring both sides in $\sqrt{4n+1} < \sqrt{n} + \sqrt{n+1}$

$\Rightarrow 4n + 1 < n + n + 1 + 2\sqrt{n(n+1)}$

$\Rightarrow 2n < 2\sqrt{n(n+1)} \Rightarrow n < \sqrt{n(n+1)}$ which is true.

II. To prove $\sqrt{n} + \sqrt{n+1} < \sqrt{4n+2}$

Squaring both sides,

$n + n + 1 + 2\sqrt{n(n+1)} < 4n+2$

$\Rightarrow 2\sqrt{n(n+1)} < 2n+1$ Squaring again

$4[n(n+1)] < 4n^2 + 1 + 4n$ or $0 < 1$ which is true

Hence $\sqrt{4n+1} < \sqrt{n} + \sqrt{n+1} < \sqrt{4n+2}$

Further to prove $[\sqrt{n} + \sqrt{n+1}] = [\sqrt{4n+1}]$ we have to prove that there is no positive integer which lies between

$\sqrt{4n+1}$ and $\sqrt{4n+2}$ or $[\sqrt{4n+1}] = [\sqrt{4n+2}]$. Using Mathematical induction.

We have to check $[\sqrt{4n+1}] = [\sqrt{4n+2}]$ for $n = 1$

$[\sqrt{5}] = [\sqrt{6}] \Rightarrow 2 = 2$, which is true

Assume for n = k (arbitrary)

i.e., $[\sqrt{4k+1}] = [\sqrt{4k+2}]$ To prove for n = k +1

To check $[\sqrt{4k+5}] = [\sqrt{4k+6}]$ since k ≥ o

Here 4k + 5 is an odd number and 4k + 6 is even number.

Their greatest integer will be different iff 4k + 6 is a perfect square that is $4k + 6 = r^2$

$\Rightarrow \quad k = \dfrac{r^2}{4} - \dfrac{6}{4}, \dfrac{6}{4}$ is not integer. But k has to be integer..

So 4k + 6 cannot be perfect square.

$\Rightarrow [\sqrt{4k+5}] = [\sqrt{4k+6}]$

By Sandwich theorem

$\Rightarrow [\sqrt{n} + \sqrt{n+1}] = [\sqrt{4n+1}]$

Q.26. Let a, b, c be positive real numbers such that b^2 - 4ac > 0 and let $\alpha_1 = c$. Prove by induction that

$\alpha_{n+1} = \dfrac{a\alpha_n^2}{\left(b^2 - 2a(\alpha_1 + \alpha_2 + ... + \alpha_n)\right)}$ **is well - defined and**

$\alpha_{n+1} < \dfrac{\alpha_n}{2}$ **for all n = 1, 2, ...** **(Here, 'well - defined' means that the denominator in the expression for α_{n+1} is not zero.) (2001 - 5 Marks)**

Ans. Sol. We have a, b, c the +ve real number s.t. $b^2 - 4ac > 0$; $\alpha_1 = c$.

$P(n) : \alpha_{n+1} = \dfrac{a\alpha_n^2}{b^2 - 2a(\alpha_1 + \alpha_2 + ... \alpha_n)}$

is well defined and $\alpha_{n+1} < \dfrac{\alpha_n}{2}, \forall n = 1, 2, ...$

For n = 1, $\alpha_2 = \dfrac{a\alpha_1^2}{b^2 - 2a\alpha_1} = \dfrac{ac^2}{b^2 - 2ac}$

Now, $b^2 - 4ac > 0 \Rightarrow b^2 - 2ac > 2ac > 0$

$\therefore$ α^2 is well defined (as denomination is not zero)

Also, $\left[\begin{array}{l} \because \quad b^2 - 2ac > 2ac \\ \Rightarrow \quad \dfrac{1}{b^2 - 2ac} < \dfrac{1}{2ac} \end{array}\right]$ $\Rightarrow$ $\dfrac{\alpha_2}{c} < \dfrac{1}{2} \Rightarrow \dfrac{\alpha_2}{\alpha_1} < \dfrac{1}{2}$

$\therefore P(n)$ is true for n = 1.

Let the statement be true for $1 \le n \le k$ i.e.,

$\alpha_{k+1} = \dfrac{a\alpha_k^2}{b^2 - 2a(\alpha_1 + \alpha_2 + \ldots \alpha_k)}$ is well defined

and $\alpha_{k+1} < \dfrac{\alpha_k}{2}$

Now, we will prove that P(k + 1) is also true

i.e. $\alpha_{k+2} = \dfrac{a\alpha_{k+1}^2}{b^2 - 2a(\alpha_1 + \alpha_2 + \ldots + \alpha_k + \alpha_{k+1})}$

well defined and $\alpha_{k+2} < \dfrac{\alpha_{k+1}}{2}$.

We have

$\alpha_1 = c, \alpha_2 < \dfrac{c}{2}, \alpha_3 < \dfrac{\alpha_2}{2} < \dfrac{c}{2^2},\ \alpha_4 < \dfrac{\alpha_3}{2} < \dfrac{c}{2^3}, \ldots$ (by IH)

Now, $(\alpha_{1+} + \alpha_2 + \ldots \alpha_k + \alpha_{k+1} < c + \dfrac{c}{2} + \dfrac{c}{2^k} + \ldots + \dfrac{c}{2^k}$

$= \dfrac{c\left(1 - \dfrac{1}{2^{k+1}}\right)}{1 - 1/2} = 2c\left(1 - \dfrac{1}{2^{k+1}}\right) < 2c$

$\therefore \quad \alpha_1 + \alpha_2 + \ldots + \alpha_{k+1} < 2c$

$\Rightarrow \quad -2a(\alpha_1 + \alpha_2 + \ldots + \alpha_{k+1}) > -4ac$

$\Rightarrow \quad b^2 - 2a(\alpha_1 + \alpha_2 + \ldots + \alpha_{k+1}) > b^2 - 4ac > 0$

$\therefore \alpha_{k+2}$ is well defined. Again by IH we have

$$\alpha_{k+1} < \frac{\alpha_k}{2} \Rightarrow 2\alpha_{k+1} < \alpha_k$$

$$\Rightarrow 4\alpha_{k+1}^2 < \alpha_k^2 \ [As \ by \ def.\alpha_{k+1},\alpha_k \ are+ve]$$

$$\Rightarrow 4\alpha_{k+1} < \frac{\alpha_k^2}{\alpha_{k+1}}$$

$$\Rightarrow 4\alpha_{k+1} < \frac{b^2 - 2a(\alpha_1 + \alpha_2 + ... + \alpha_k)}{a}$$

$$\Rightarrow 4a\alpha_{k+1} < b^2 - 2a(\alpha_1 + \alpha_2 + ... + \alpha_k)$$

$$\Rightarrow 2a\alpha_{k+1} < b^2 - 2a(\alpha_1 + \alpha_2 + ... + \alpha_k + \alpha_{k+1})$$

$$\Rightarrow \frac{a\alpha_{k+1}^2}{b^2 - 2a(\alpha_1 + \alpha_2 + ... + \alpha_{k+1})} < \frac{1}{2}$$

$$\Rightarrow \frac{a\alpha_{k+1}}{b^2 - 2a(\alpha_1 + \alpha_2 + ... + \alpha_{k+1})} < \frac{\alpha_{k+1}}{2}$$

$$\Rightarrow \alpha_{k+2} < \frac{\alpha_{k+1}}{2}$$

$\therefore$ P(k+1) is also true.

Thus by the Principle of Mathematical Induction the Statement P(n) is true $\forall \ n \in N$.

Q.27. Use mathematical induction to show that $(25)^{n+1} - 24n + 5735$ is divisible by $(24)^2$ for all n = 1, 2, ... (2002 - 5 Marks)

Ans. Sol. Let P(n) : $(25)^{n+1} - 24n + 5735$ For n = 1.

P(1) : $625 - 24 + 5735 = 6336 = (24)2 \times (11)$,

which is divisible by 242.

Hence P(1) is true Let P(k) be true, where $k \geq 1$

$$\Rightarrow (25)^{k+1} - 24k + 5735$$

$= (24)^2 \lambda$ where $\lambda \in N$

For n = k + 1,

P (k + 1) : (25) k + 2 – 24 (k + 1) + 5735

= 25 [(25) k + 1 – 24k + 5735]

+ 25.24.k – (25) (5735) + 5735 – 24 (k + 1)

= 25 $(24)^2 \lambda$ + $(24)^2$ k – 5735 × 24 – 24

= 25 $(24)^2 \lambda$ + $(24)^2$ k – (24) (5736)

= 25 $(24)^2 \lambda$ + $(24)^2$ k – $(24)^2$ (239),

= $(24)^2$ [25 λ + k – 239]

which is divisible by (24)2.

Hence, by the method of mathematical induction result is true $\forall$ n$\in$ N .

Q.28. Prove that (2003 - 2 Marks)

$$2^k \binom{n}{0}\binom{n}{k} - 2^{k-1}\binom{n}{1}\binom{n-1}{k-1}$$

$$+ 2^{k-2}\binom{n-2}{k-2} - \dots (-1)^k \binom{n}{k}\binom{n-k}{0} = \binom{n}{k}.$$

Ans. Sol. To prove that $2^k \, {}^nC_0 {}^nC_k - 2^{k-1} \, {}^nC_1 {}^{n-1}C_{k-1} + 2^{k-2} \, {}^nC_2 {}^{n-2}C_{k-2}$

$-\dots + (-1)^k \, {}^nC_k {}^{n-k}C_0 = {}^n C_k$

LHS of above equation can be written as

$$\sum_{r=0}^{k} (-1)^r 2^{k-r} \, {}^nC_r {}^{n-r}C_{k-r}$$

$$= \sum_{r=0}^{k} (-1)^r 2^{k-r} \frac{n!}{r!(n-r)!} \frac{(n-r)!}{(k-r)!(n-k)!}$$

$$= \sum_{r=0}^{k} (-1)^r 2^{k-r} \frac{n!k!}{r!k!(n-k)!(k-r)!}$$

$$= \sum_{r=0}^{k} (-1)^r \frac{2^k}{2^r} \cdot \frac{n!}{k!(n-k)!} \cdot \frac{k!}{r!(k-r)!}$$

$$= 2^k \ {}^nC_k \sum_{r=0}^{k} (-1/2)^r \frac{k!}{r!(k-r)!}$$

$$= 2^k \ {}^nC_k \sum_{r=0}^{k} {}^kC_r (-1/2)^r = 2^k \ {}^nC_k (1-1/2)^k$$

$$= 2^k \ {}^nC_k \frac{1}{2^k} = {}^nC_k \text{ R.H.S. Hence Proved}$$

Q.29. A coin has probability p of showing head when tossed. It is tossed n times. Let p_n denote the probability that no two (or more) consecutive heads occur. Prove that $p_1=1$, $p_2=1-p^2$ and $p_n=(1-p). \ p_{n-1} + p(1-p) \ p_{n-2}$ for all $n \geq 3$. Prove by induction on n, that $p_n = A\alpha^n + B\beta^n$ for all $n \geq 1$, where a and b are the roots of quadratic equation
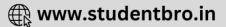
$$x^2 - (1-p)x - p(1-p) = 0 \text{ and } A = \frac{p^2 + \beta - 1}{\alpha\beta - \alpha^2}, \ B = \frac{p^2 + \alpha - 1}{\alpha\beta - \beta^2}.$$

Ans. Sol. We have $\alpha + \beta = 1 - p$ and $\alpha\beta = -p(1-p)$

For $n = 1$, $p_n = p_1 = 1$

Also, $A\alpha^n + B\beta^n = A\alpha + B\beta = \frac{(p^2 + \beta - 1)\alpha}{\alpha\beta - \alpha^2}$

$$+ \frac{(p^2 + \alpha - 1)\beta}{\alpha\beta - \beta^2} = \frac{p^2 + \beta - 1}{\beta - \alpha} + \frac{p^2 + \alpha - 1}{\alpha - \beta}$$

$$= \frac{p^2 + \beta - 1 - p^2 - \alpha + 1}{\beta - \alpha} = \frac{\beta - \alpha}{\beta - \alpha} = 1$$

For $n = 2$, $p_2 = 1 - p^2$

Also, $A\alpha^n + B\beta^n = A\alpha^2 + B\beta^2$

$$= \frac{(p^2 + \beta - 1)\alpha^2}{\alpha\beta - \alpha^2} + \frac{(p^2 + \alpha - 1)\beta^2}{\alpha\beta - \alpha^2}$$

which is true for n = 2

Now let result is true for k < n where n ≥ 3.

$$P_n = (1-p)P_{n-1} + p(1-p)P_{n-2}$$

$$= (1-p)(A\alpha^{n-1} + B\beta^{n-1}) + p(1-p)(A\alpha^{n-2} + B\beta^{n-2})$$

$$= A\alpha^{n-2}\{(1-p)\alpha + p(1-p)\} + B\beta^{n-2}\{(1-p)\beta - p(1-p)\}$$

$$= A\alpha^{n-2}\{(\alpha+\beta)\alpha - \alpha\beta\}$$

$$\vdash B\beta^{n-2}\{(\alpha+\beta)\beta - \alpha\beta\} \text{ by } (1)$$

$$= A\alpha^{n-2}\{\alpha^2 + \beta\alpha - \alpha\beta\} + B\beta^{n-2}\{\alpha\beta + \beta^2 - \alpha\beta\}$$

$$= A\alpha^{n-2}(\alpha^2) + B\beta^{n-2}(\beta^2) = A\alpha^n + B\beta^n$$

This is true for n. Hence by principle of mathematical induction, the result holds good for all n ∈ N .

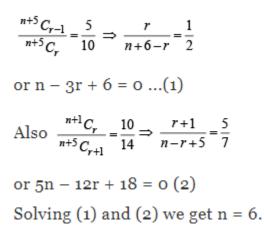# Integar Type ques of Mathematical Induction and Binomial Theorem

Q.1. The coefficients of three consecutive terms of $(1 + x)^{n+5}$ are in the ratio 5 : 10 : 14. Then n = (JEE Adv. 2013)

Ans. (6)

Sol. Let the coefficients of three consecutive term s of $(1 + x)^{n+5}$

be $^{n+5}C_{r-1},\ ^{n+5}C_r,\ ^{n+5}C_{r+1}$,

then we have $^{n+5}C_{r-1} : ^{n+5}C_r : ^{n+5}C_{r+1} = 5 : 10 : 14$

$$\frac{^{n+5}C_{r-1}}{^{n+5}C_r} = \frac{5}{10} \Rightarrow \frac{r}{n+6-r} = \frac{1}{2}$$

or $n - 3r + 6 = 0$ ...(1)

Also $\dfrac{^{n+1}C_r}{^{n+5}C_{r+1}} = \dfrac{10}{14} \Rightarrow \dfrac{r+1}{n-r+5} = \dfrac{5}{7}$

or $5n - 12r + 18 = 0$ (2)
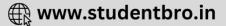
Solving (1) and (2) we get n = 6.

Q. 2. Let m be th e sm allest posi tive i n teger such th at th e coefficient of x^2 in the expansion of $(1 + x)^2 + (1 + x)^3 + ... + (1 + x)^{49} + (1 + mx)^{50}$ is $(3n + 1)\ ^{51}C_3$ for some positive integer n. Then the value of n is (JEE Adv. 2016)

Ans. (5)

Sol. $(1 + x)^2 + (1 + x)^3 + ... + (1 + x)^{49} + (1 + mx)^{50}$

$$= (1+x)^2 \left[\frac{(1+x)^{48} - 1}{(1+x) - 1} \right] + (1+mx)^{50}$$

$$= \frac{1}{x} \left[(1+x)^{50} - (1+x)^2 \right] + (1+mx)^{50}$$

Coeff. of x^2 in the above expansion = Coeff. of x^3 in $(1 + x)^{50}$ + Coeff. of x^2 in $(1 + mx)^{50}$

$\Rightarrow$ $^{50}C_3 + {}^{50}C_2\, m^2$

$\therefore (3n + 1)\, {}^{51}C_3 = {}^{50}C_3 + {}^{50}C_2\, m^2$

$\Rightarrow (3n+1) = \dfrac{{}^{50}C_3}{{}^{51}C_3} + \dfrac{{}^{50}C_2}{{}^{51}C_3} m^2$

$\Rightarrow 3n + 1 = \dfrac{16}{17} + \dfrac{1}{17} m^2 \Rightarrow n = \dfrac{m^2 - 1}{51}$

Least positive integer m for which n is an integer is m = 16 and then n = 5